

Periodically driven ergodic and many-body localized quantum systems

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We study dynamics of isolated quantum many-body systems under periodic driving. We consider a driving protocol in which the Hamiltonian is switched between two different operators periodically in time. The eigenvalue problem of the associated Floquet operator maps onto an effective hopping problem in energy space. Using the effective model, we establish conditions on the spectral properties of the two Hamiltonians for the system to localize in energy space. We find that ergodic systems always delocalize in energy space and heat up to infinite temperature, for both local and global driving. In contrast, many-body localized systems with quenched disorder remain localized at finite energy. We argue that our results hold for general driving protocols, and discuss their experimental implications.

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Introduction. Quantum systems coupled to time-varying external fields are ubiquitous in nature. They exhibit many interesting phenomena including the laser, the maser, electron-spin resonance and nuclear magnetic resonance (NMR) [1, 2]. The experimental developments in ultra-cold atomic or molecular gases and trapped ions in the last two decades have taken us beyond the few-atom systems into the regime of isolated *interacting* systems, whose quantum dynamics reveals novel aspects of thermalization, transport and non-linear response [3, 4].

Periodically driven systems can exhibit non-trivial steady states, even in the non-interacting limit [5–9]. An illustrative system is the kicked quantum rotor, which is dynamically localized or Anderson localized in momentum space [5–8]. For special types of driving, a universal localization-delocalization transition occurs in momentum space. This has been observed experimentally [7, 8]. Periodic driving can also be used to control the band structure of non-interacting particles, e.g., to induce topologically non-trivial states [10–13].

In this article, we explore the nature of the steady states of periodically driven many-body systems with local interactions. If the driving were to lead to a thermal steady state, its temperature is expected to be infinite, as there are no conserved quantities when a generic Hamiltonian becomes time-dependent. However, recently D’Alessio and Polkovnikov [14] raised the interesting possibility of localization in energy space, inspired by a numerical study. This poses an important question: when does a driven system heat up to infinite temperature and when does it dynamically localize?

Here we answer this question by providing a criterion for delocalization in energy space. We consider a simple driving of the Hamiltonian $\hat{H}(t)$ in every period T :

$$\hat{H}(t) = \begin{cases} \hat{V} & 0 < t < T_1 \\ \hat{H}_0 & T_1 < t < T_0 + T_1 \end{cases} \quad (1)$$

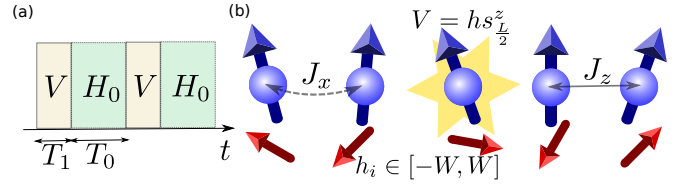


FIG. 1. (Color online) (a) A general driving protocol consists of applying an operator \hat{V} for time T_1 , followed by \hat{H}_0 applied for time T_0 . (b) Example of a 1d XXZ spin chain studied numerically (blue arrows). The Hamiltonian H_0 contains nearest-neighbor hopping and interactions, in the presence of random z -field (red). Driving is performed by the local operator $\hat{V} = hs_{L/2}^z$ applied to the middle spin.

where T_0 (T_1) is the time over which \hat{H}_0 (\hat{V}) is applied ($T_0 + T_1 = T$), see Fig. 1(a). This protocol can be viewed as a many-body generalization of the kicked rotor model. For *local* \hat{V} , we show that the competition between the typical matrix elements of $\tan(\hat{V}T_1/2)$ between eigenstates of H_0 and the typical energy spacings determines whether or not the system will heat up to infinite temperature. We then apply our criterion to two general classes of driven many-body systems. The first class is ergodic systems – i.e., systems that act as their own heat bath and satisfy the eigenstate thermalization hypothesis (ETH) [15–17]. We show that ergodic systems generally heat up to infinite temperature under driving. The second class is *many-body localized* (MBL) systems with quenched disorder that are known to be non-ergodic [18–22]. Under local driving, we show that MBL systems retain memory of their initial state and never reach infinite temperature. We support our analytical results with a numerical study of the driven XXZ spin-1/2 chain in random z -fields, Fig. 1(b). This model exhibits both the ergodic and the MBL phase in the absence of driving [19].

Our results bear on the structure of the eigenstates

of the Floquet operator – the evolution operator for one time period – as well as on the locality of the so-called Floquet Hamiltonian \hat{H}_F [23], defined as $\hat{F} = e^{-i\hat{H}_F T}$. In the ergodic case, each Floquet eigenstate describes an infinite-temperature state for local observables. It involves the eigenstates of \hat{H}_0 at all energy densities, and has a non-zero participation ratio (PR) in this basis in the thermodynamic limit. The Floquet Hamiltonian is therefore *non-local*. In contrast, in the MBL case, each Floquet eigenstate overlaps with a vanishingly small fraction of the eigenstates of \hat{H}_0 in the thermodynamic limit. Further, the overlap is substantial with only a few eigenstates that have similar energy. The Floquet Hamiltonian is local, and is itself MBL.

From Floquet to a hopping problem. The Floquet operator for the driving protocol in Eq. (1) is given by:

$$\hat{F} = e^{-i\hat{H}_0 T_0} e^{-i\hat{V} T_1}. \quad (2)$$

The eigenstates of \hat{F} completely determine the stroboscopic evolution of the system. Below, we map the eigenvalue problem of \hat{F} onto a hopping problem, similar to the kicked rotor model in Ref. [6]. The lattice sites of the hopping problem represent eigenstates of \hat{H}_0 , while \hat{V} induces hopping between different sites.

The Floquet operator is unitary. Its spectrum is:

$$\hat{F}|\psi_i\rangle = e^{-i\hat{H}_F T}|\psi_i\rangle = e^{-i\omega_i T}|\psi_i\rangle, \quad i = 1, \dots, \mathcal{M}, \quad (3)$$

where \mathcal{M} is the dimensionality of the (many-body) Hilbert space (e.g., $\mathcal{M} = 2^N$ for the system of N 1/2-spins considered below), $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, and the ω_i are the quasi-energies of the Floquet Hamiltonian \hat{H}_F . As quasienergies ω_i are defined modulo $2\pi/T$, the Floquet Hamiltonian is not unique.

Determining the spectrum of \hat{F} in a many-body system is generally hard, due to \hat{F} being highly non-local. To circumvent this difficulty, let us provide an explicit mapping to a local Hamiltonian problem. Rewrite $e^{-i\hat{V} T_1}$ in terms of a Hermitian operator \hat{G} as:

$$e^{-i\hat{V} T_1} = \frac{1 + i\hat{G}}{1 - i\hat{G}}, \quad \hat{G} = -\tan \frac{\hat{V} T_1}{2}. \quad (4)$$

In general \hat{G} is not spatially local. If however \hat{V} is local (that is, acts non-trivially only on a small number of spatial degrees of freedom), then \hat{G} is also local. Defining $|\chi_i\rangle \equiv (1 - i\hat{G})^{-1}|\psi_i\rangle$, Eq. (3) becomes:

$$\left(\tan \frac{\hat{H}_0 T_0 - \omega_i T}{2} - \hat{G} \right) |\chi_i\rangle = 0. \quad (5)$$

Let us view the eigenbasis of \hat{H}_0 , labeled by $|\alpha\rangle$, as sites in a lattice. Solving Eq. (5) is equivalent to finding the zero-energy eigenstate of a hopping problem on this lattice, where $\tan \frac{E_\alpha T_0 - \omega_i T}{2}$ plays the role of an on-site energy

on site α , and $G_{\alpha\beta}$ is the hopping amplitude between sites α and β . The properties of the hopping matrix elements, which are different in the ergodic and MBL phases [19, 24–26], thus determine the structure of the Floquet eigenstates.

Delocalization in the hopping problem implies heating. We imagine preparing the system at $t = 0$ in a low-energy eigenstate of \hat{H}_0 , $|\varphi_0\rangle = |\alpha_0\rangle$. The stroboscopic evolution at times $t = NT$ follows from the expansion of $|\varphi_0\rangle$ in the Floquet eigenbasis:

$$|\varphi_N\rangle = \hat{F}^N |\varphi_0\rangle = \sum_i A_{\alpha_0 i} e^{-i\omega_i NT} |\psi_i\rangle, \quad (6)$$

where $A_{\alpha i} = \langle\psi_i|\alpha\rangle$. At long times, the time-averaged density matrix is $\hat{\rho}_\infty = \sum_i |A_{\alpha_0 i}|^2 |\psi_i\rangle\langle\psi_i|$. The nature of the eigenstates $|\psi_i\rangle$ determines the steady state as $t \rightarrow \infty$. If each $|\psi_i\rangle$ is delocalized in the eigenbasis of \hat{H}_0 , then each $|\psi_i\rangle$ corresponds to an infinite temperature state. The entire density matrix, $\hat{\rho}_\infty$, describes a system at infinite temperature in this case. If on the other hand the $|\psi_i\rangle$ are localized in the eigenbasis of \hat{H}_0 , then, depending on $A_{\alpha_0 i}$, $\hat{\rho}_\infty$ describes a system at different energies.

To characterize the energy absorbed under driving, we introduce a dimensionless energy at each $t = NT$:

$$Q_N = \frac{\langle\psi_N|\hat{H}_0|\psi_N\rangle - E_0}{E_{T=\infty} - E_0}, \quad (7)$$

where $E_{T=\infty} = \text{Tr}(\hat{H}_0)/\mathcal{M}$ is the average energy at infinite temperature, and E_0 is the energy at $t = 0$. Ergodic and MBL cases are distinguished by $Q_N \rightarrow 1$ and $Q_N \rightarrow 0$ as $N \rightarrow \infty$, respectively.

Local driving. Suppose \hat{V} is local. Then, $\hat{G} = -\tan \frac{\hat{V} T_1}{2}$ is also local and its spectrum is discrete. \hat{G} has a finite operator norm, $\|\hat{G}\| \leq C$, except at special values of T_1 when $\eta_i T_1 = (2n+1)\pi$, where n is an integer and η_i are the eigenvalues of \hat{V} .

We expect the zero-energy states $|\chi_i\rangle$ in Eq. (5) to be localized in the eigenbasis of \hat{H}_0 when the typical hopping amplitude $G_{\alpha\beta}$ is much smaller than the level spacing $\Delta\lambda$. At quasi-energy ω , the on-site energies $\lambda_\alpha(\omega) = \tan \frac{E_\alpha T_0 - \omega T}{2}$ have a distribution $P(\lambda) \propto \frac{1}{1+\lambda^2}$ if the E_α are distributed uniformly. As we are searching for a zero-energy solution, we focus on $|\lambda_\alpha(\omega)| \lesssim 1$. From the form of $P(\lambda)$, we see that the level spacing for $|\lambda_\alpha(\omega)| \lesssim 1$ is $\Delta\lambda \approx \Delta E \sim \frac{1}{\mathcal{M}}$, where ΔE is the level spacing in energies of \hat{H}_0 . The criterion for localization thus takes the following form:

$$|G_{\alpha\beta}| = |\langle\alpha|\hat{G}|\beta\rangle| \ll \frac{1}{\mathcal{M}}. \quad (8)$$

Two comments are in order. First, the sites that we have not included have energy $\lambda_\alpha(\omega) \gg 1$, as well as level spacing $\Delta\lambda \gg 1$. Such states form a small fraction of the Hilbert space in the thermodynamic limit and

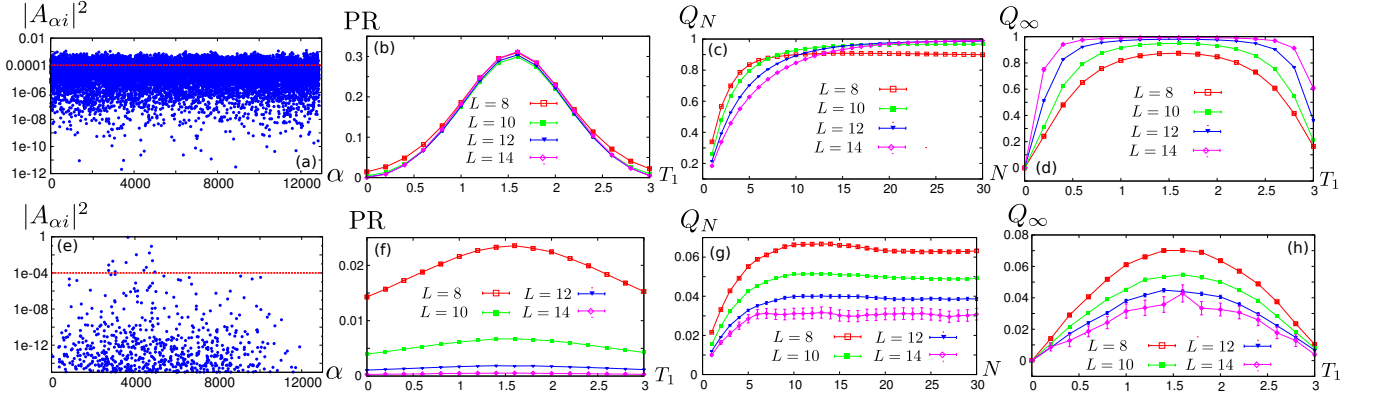


FIG. 2. (Color online) Top row [(a)-(d)]: Energy delocalization and heating to infinite temperature in the ergodic phase of the XXZ model ($T_0 = 7, W = 0.5$). Bottom row [(e)-(h)]: Energy localization and absence of heating in the MBL phase ($T_0 = 7, W = 8$). Panels (a),(e): squared overlap $|A_{\alpha i}|^2$ of the Floquet eigenstate $|\psi_i\rangle$ with eigenstates $|\alpha\rangle$ of \hat{H}_0 , ordered by energy, for a fixed disorder realization and $T_1 = 1.5$. In the ergodic case (a), $|A_{\alpha i}|^2$ are nearly uniformly spread over all eigenstates α (dashed line), while in the MBL case (e), the overlap is non-zero only for a few eigenstates with similar energies. Panels (b), (f): disorder-averaged PR, Eq. (10), vs. T_1 . As L is increased, PR remains finite or drops as $1/\mathcal{M}$, in the ergodic and MBL case, respectively. Panels (c), (g): disorder-averaged Q_N (Eq. 7) vs. the number of driving cycles N ($T_1 = 1.5$), for evolution starting from the ground state. Panels (d), (h): disorder-averaged saturation value Q_∞ vs. T_1 for different L . In the ergodic case (d), Q_∞ sharply approaches 1 as $L \rightarrow \infty$ for any T_1 , signaling generic heating to infinite temperature. For the MBL case (h), $Q_\infty \ll 1$ and decreases with L , indicating that the system absorbs energy locally. Number of disorder averages is 2×10^4 ($L = 8, 10$) and $\sim 10^3$ ($L = 12, 14$).

are extremely off-resonant; we are thus well-justified in ignoring them. Second, using Eq. (5), the Floquet eigenstates $|\psi_i\rangle$ are localized (delocalized) if $|\chi_i\rangle$ are localized (delocalized).

Ergodic systems. The condition (8) does not hold for ergodic systems even for arbitrarily weak driving. To see why, note that eigenstates $|\alpha\rangle$ in the ergodic phase are like random vectors in the basis of product states that satisfy the ETH. The typical matrix element of *any* local operator between two random vectors is:

$$|G_{\alpha\beta}| \sim \frac{\|G\|}{\sqrt{\mathcal{M}}} \gg \frac{1}{\mathcal{M}}. \quad (9)$$

The scaling with the size of the Hilbert space is independent on the driving strength $\|\hat{G}\|$. Thus, in the thermodynamic limit, the hopping problem is always in the delocalized phase.

In the delocalized phase, the Floquet eigenstates are superpositions of $|\alpha\rangle$ with macroscopically different physical energies E_α (but with $\lambda_\alpha(\omega) \lesssim 1$). More precisely, the participation ratio (PR) of the Floquet eigenstate $|\psi_i\rangle$ in the basis of eigenstates α , defined as:

$$\text{PR} = \frac{1}{\mathcal{M}} \left(\sum_{\alpha} |A_{\alpha i}|^4 \right)^{-1} \quad (10)$$

remains finite as $\mathcal{M} \rightarrow \infty$. The fraction of infinite-temperature states in the Hilbert space also increases as $\mathcal{M} \rightarrow \infty$; thus individual Floquet eigenstates describe infinite-temperature states of the system. Below, we will numerically confirm this behavior of the PR. We will also show that Q_N (Eq. (7)) tends to one as $N \rightarrow \infty$.

Many-body localized systems. The situation is very different for MBL states. The typical matrix elements of local operators in the MBL phase decay exponentially with the system size, but fall off faster than the level spacing [27]:

$$G_{\alpha\beta} \sim \|G\| e^{-L/\xi} \ll \frac{1}{\mathcal{M}}, \quad (11)$$

where ξ is the “many-body” localization length (possibly different from the localization length of single-particle operators). The MBL phase is characterized by local integrals of motion with exponentially decaying tails [21, 22]. Eigenstates α, β that have nearly the same energy typically differ by the values of the local integrals of motion a distance $\sim L$ away from the support of local operator G . The probability of changing the value of a remote integral of motion decays exponentially with the distance, which explains the origin of the result (11).

The criterion for localization in (8) is satisfied in the MBL phase. Hopping \hat{G} only significantly mixes a few eigenstates with a similar structure away from the support of \hat{G} (that is, with the same values of the local integrals of motion distance $x \gtrsim \xi$ away). This implies that the system can absorb energy only locally, in the vicinity of the driving.

The structure of the Floquet eigenstates in the localized phase is thus very different from the ergodic case. First, their PR exponentially approaches zero with system size as $1/\mathcal{M}$. Second, each eigenstate involves states $|\alpha\rangle$ which are close in energy (since only states which differ locally are mixed by the operator \hat{G}). When the

system is prepared in an initial state with some energy, the amount of absorbed energy at long times is independent of system size, and the system never heats up to infinite temperature ($Q_\infty = 0$ as $\mathcal{M} \rightarrow \infty$).

Numerical simulations. We consider the XXZ spin-1/2 chain with L sites and open boundary conditions:

$$\hat{H}_0 = J_x \sum_i (s_i^x s_{i+1}^x + s_i^y s_{i+1}^y) + J_z \sum_i s_i^z s_{i+1}^z + \sum_i h_i s_i^z, \quad (12)$$

where fields h_i are independent random variables drawn from the uniform distribution $[-W, W]$, and we fix $J_x = J_z = 1$. The model (12) exhibits both ergodic and MBL phases as a function of disorder strength W , with the transition at $W_* \approx 3$ [19]. The system is driven by

$$\hat{V} = h s_{L/2}^z \quad (13)$$

acting on the middle spin (we assume $h = 2$). Fig. 2 summarizes the numerical results regarding stroboscopic evolution and Floquet eigenstates, in ergodic and MBL regimes. In Fig. 2 we fix $T_0 = 7$, chosen to minimize finite-size effects and thus satisfy $T_0 \Delta \gg 1$, where Δ is the bandwidth. The ergodic and MBL cases correspond to disorder $W = 0.5$ and $W = 8$, respectively. Number of disorder averages is 2×10^4 ($L = 8, 10$) and $\sim 10^3$ ($L = 12, 14$).

In Figs. 2(a),(e), we first illustrate the structure of a typical Floquet eigenstate $|\psi_i\rangle$ for a fixed disorder realization and $T_1 = 1.5$. We plot $|A_{\alpha i}|^2$ as function of α , ordered by energy E_α . In the ergodic case (a), the Floquet eigenstate is delocalized and has non-zero overlap with states at very different energies. In contrast, in the MBL case (e), an individual Floquet eigenstate has sizable overlap only with those $|\alpha\rangle$ that are close in energy.

The difference between Floquet eigenstates is further revealed in the behaviour of PR, Figs. 2(b), (f). Disorder-averaged PR, plotted as a function of T_1 for different system sizes L , shows that Floquet eigenstates occupy a finite fraction of the Hilbert space in the thermodynamic limit when the system is ergodic (b), while PR decreases as $1/\mathcal{M}$ in the MBL case (f).

The energy absorbed after N cycles, Q_N , when the system is initially prepared in the ground state of H_0 , is shown in Figs. 2(c),(g). In the thermodynamic limit, Q_N approaches 1 for ergodic systems (c), while in the MBL case (f) Q_N is much smaller than 1 and decreases with system size. Similarly, the saturated value Q_∞ in the ergodic phase (Fig. 2(d)) tends to 1 as L is increased. Note that, for the system sizes studied here, $Q_\infty \ll 1$ for small T_1 , due to the small norm of the operator G . However, Q_∞ monotonically increases as a function of L , and is likely to reach 1 in the thermodynamic limit, even for arbitrarily small values of T_1 , suggesting that ergodic systems generally heat up to infinite temperature. On the other hand, in the MBL case, Fig. 2(h), Q_∞ is much smaller than one for all T_1 , and decays as $1/L$.

These features reflect the local absorption of energy in the system. The different finite-size scaling of the absorbed energy for the ergodic and MBL cases is shown in Fig. 3.

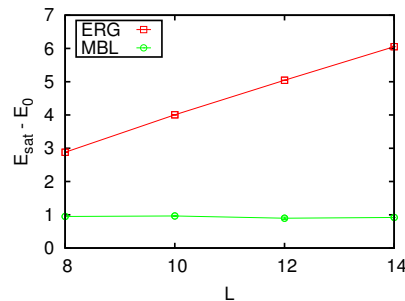


FIG. 3. (Color online) Absorbed energy $E_{\text{sat}} - E_0 = \text{Tr}(\rho_\infty H_0) - E_0$ as a function of system size L for ergodic (red) and MBL (green) phases. In the former case, absorbed energy is extensive and scales linearly with L , while in the latter case energy is absorbed only locally and its value is independent of L .

Finally, we have also numerically studied the level statistics of the Floquet operator, characterized by the parameter [19] $r = \langle \min(\Delta\omega_i, \Delta\omega_{i+1}) / \max(\Delta\omega_i, \Delta\omega_{i+1}) \rangle$, where $\Delta\omega_i = \omega_i - \omega_{i-1}$ and ω_i is chosen to lie in the interval $[0, 2\pi)$. In the ergodic phase, $r \approx 0.53$, reflecting level repulsion and the circular orthogonal ensemble [28], while in the MBL case, $r \approx 0.386$, consistent with the Poisson statistics.

Concluding remarks. We have shown that the effect of periodic local driving is very different in ergodic and many-body localized systems. Driven ergodic systems heat up to infinite temperature, and their Floquet eigenstates are delocalized in energy space. MBL systems, on the other hand, absorb energy only locally, and provide an example of dynamical localization.

A few remarks are in order. First, we have verified that our results hold for other choices of (local) \hat{V} and \hat{H}_0 . In particular, we believe that our conclusions hold for harmonic driving $\hat{H}(t) = \hat{H}_0 + \hat{V} \cos(\Omega t)$. Second, our approach can be extended to the case of *global* driving – that is, when \hat{V} is a sum of local terms. The mapping of the Floquet problem onto a hopping problem is unchanged. However, the hopping operator \hat{G} is no longer bounded as $L \rightarrow \infty$, which can only help with delocalization. This also agrees with the very recent work of D’Alessio and Rigol [28] who numerically showed that the level statistics of the Floquet operator is described by the circular ensemble, and argued that such statistics signals heating up to infinite temperature. Thus, globally driven ergodic systems are also expected to heat to infinite temperature, and to have delocalized Floquet eigenstates. The case of globally driven MBL systems, on the other hand, is more intricate and deserves a sep-

arate study [29]. Finally, we note that our results can be tested in systems of ultra-cold atoms in optical lattices with existing experimental technology. Periodically driven systems of interacting bosons and fermions have been realized (see, e.g. [30] and [4] for a review). We also note that optical lattices with tunable disorder have been realized, and Anderson localization in them was observed [31]. By tuning the interactions [32–34], it should be possible to realize an MBL phase and study its response under driving.

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Note added. While this paper was being finalized, we learned of a recent numerical study [35], which finds that globally driven ergodic systems heat up to infinite temperature. This agrees with Ref. [28] and with the part of our results that concern the ergodic systems.

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